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A computational procedure for the implementation of equivalent linearization in finite element analysis

H. H. Emam, H. J. Pradlwarter and G. I. Schuëller*,†

Institute of Engineering Mechanics, Leopold-Franzens University, Innsbruck, Austria

SUMMARY

This paper deals with the practical implementation of the statistical equivalent linearization method (EQL) in conjunction with general FE-analysis to evaluate non-linear structural response under random excitation. A computational procedure is presented which requires the non-linear part of the system to be subdivided into suitable sub-domains (elements). Each element is independently linearized using only a minimum number of co-ordinates. A local co-ordinate system is introduced using linear transformations of the global (master) degrees of freedom. Restoring forces and non-linear constitutive laws are defined by the local co-ordinates of each element. The linearization coefficients are further transformed back to establish the global linearized system. The procedure has, on one hand, the ability to use any desired linearization criterion and, on the other hand, it can be combined with highly developed procedures to determine the response of arbitrary large FE-models. To illustrate the applicability of the procedure, two different non-linear systems are analysed under bi-directional earthquake excitation. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: statistical equivalent linearization; finite element; stochastic response; non-linear systems

1. INTRODUCTION

The development of various methods to evaluate the stochastic response of non-linear systems under general type of random loading has been accelerated over the last few decades. The main target of these methods would be their implementation within finite element (FE)-based structural analyses. Since no exact analytical procedure can deal with general arbitrary non-linear systems, most of the research efforts focused on developing approximate methods [1] to solve the non-linear equations of motion. The available analytical and numerical methods depend heavily on many aspects, e.g. the degree of freedom (DOF) of the structural model, the type and degree of its non-linear behaviour, the random excitation model and the distribution of the resulting stochastic response, respectively. As an example, procedures based on solving the Fokker–Planck equation

*Correspondence to: G.I. Schuëller, Institute of Engineering Mechanics, Leopold-Franzens University, Innsbruck A-6020, Austria.
†E-mail: Mechanik@uibk.ac.at
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[2,3], closure techniques [4] and stochastic averaging [5,6] appear to be the most efficient especially for structural models with very few DOFs whereas for other analytical procedures, e.g. the Perturbation method [7], are limited to systems with weak non-linear behaviour.

In view of these restrictions, the method of statistical equivalent linearization (EQL) [8,9,10] is considered as the most attractive analytical tool especially for general multi-DOF systems represented by FE-models as often encountered in practical applications [11]. Detailed reviews of the method can be found elsewhere [12,13].

In principle, the method of statistical equivalent linearization is based on the simple idea to replace a given system of non-linear equations by a 'statistically equivalent' linear one such that the mean-square error between the actual non-linear and linearized systems is minimized in a statistical sense. This approach has the essential advantage that well-developed linear random vibration theory and the associated algorithms can be efficiently utilized. However, in some cases of strong non-linearity, one has to pay attention to the obtained results since the non-linear behaviour of the system might not be suitably simplified by a linearized one. In this respect, numerical methods such as Monte Carlo simulations can estimate the stochastic response of general non-linear MDOF-systems without any restriction. In many practical applications, however, the accurate response is quite near to the obtained 'statistical equivalent' linear response. This is, for example, the common case of structures which globally behave in a linear range while some parts react non-linearly, e.g. due to friction, plasticity, degradation, non-linear contacts, etc. Although the non-linearity might be locally considered as strongly non-linear, its effect on the overall response is often not so dramatic. Typical examples are hysteretic cycles of inelastic deformation under severe natural hazard loading within certain cross-sections [14,15,16,17], which affect the energy dissipation, i.e. the modal damping ratios of the linearized structures and to a much smaller extent the mode shapes.

From the literature on statistical linearization, one can find that most of the available articles discuss the linearization technique of different non-linear mechanisms and/or various criteria to establish 'statistically equivalent' linear systems. Little work deals with the iterative stationary and non-stationary response evaluation of the linearized structure. Even less work [18,19] exists, however, which deals with the *practical* implementation of statistical linearization when applied to general linear FE-models. In this paper, the attention is directed to this practical aspect which is important for FE-models since these models are much more complex in analysis than the generally used shear-beam models. In this context a computational procedure which makes use of the statistical linearization method in accordance with FE-analysis is presented. The non-linear part of the system is descretized into suitable elements where each non-linear element is linearized independently by any desired criterion. To avoid difficulties associated with the linearization scheme, it is suggested to perform the linearization in a local co-ordinate system with a minimum number of co-ordinates. This can be achieved by introducing suitable transformations from the global to the local co-ordinate system. After establishing the linearized elements in minimal local co-ordinates, these transformations are then further used to obtain the linearized elements in the global co-ordinate system. Applying this procedure, statistical linearization can be used in a straightforward manner for any FE-model.

One of the most attractive advantages of the proposed procedure is the drastically reduced computational efforts required particularly when compared to direct Monte Carlo simulation procedures. The procedure is numerically demonstrated by analysing two different MDOF-systems with frictional devices exhibiting one- or two-dimensional hysteretic behaviour under bi-directional earthquake excitation.

This method is already embedded in a respective software environment [20] to be readily used in the engineering practice.

2. NON-LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS

The following multi-degree of freedom (MDOF) equation of motion describes the (generalized) displacement response of a non-linear structure with n degrees of freedom

$$\mathbf{M} \cdot \ddot{\mathbf{u}} + \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{F}(t) \tag{1}$$

where $\mathbf{M}$ is an $n \times n$ generalized mass matrix, $\mathbf{f}(\mathbf{u}, \dot{\mathbf{u}})$ is an n-dimensional non-linear vector function of generalized displacements $\mathbf{u}$ and velocities $\dot{\mathbf{u}}$ and $\mathbf{F}(t)$ stands for the generalized excitation force vector which is modelled as filtered white noise:

$$\mathbf{f}(t) = \mathscr{E} \cdot \mathbf{z}(t) \tag{2}$$

where $\mathscr{E}$ denotes a constant matrix and $\mathbf{z}(t)$ is a vector of linear filter parameters governed by the linear differential equation:

$$\dot{\mathbf{z}}(t) = \mathscr{F} \cdot \mathbf{z}(t) + \mathbf{w}(t) \tag{3}$$

In the above relation, the matrix $\mathscr{F}$ defines the filter and $\mathbf{w}(t)$ is a vector comprising white noise components.

In engineering applications, it is quite often the case where the non-linear function $\mathbf{f}(\mathbf{u}, \dot{\mathbf{u}})$ can be written as a sum of a linear damping term (as a function of the damping matrix $\mathbf{D}$) and a non-linear restoring force term in the particular form

$$\mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{D} \cdot \dot{\mathbf{u}} + \mathbf{r}(t) \tag{4}$$

Applying a finite element discretization scheme, $\mathbf{r}(t)$ can be subdivided into suitable non-linear elements. For each element, denoted by '^{e}', a non-linear function $^e\mathbf{g}$ of the complete state of the element can be represented as

$$^e\mathbf{r}(t) = {}^e\mathbf{g}[{}^e\mathbf{u}, {}^e\dot{\mathbf{u}}, {}^e\mathbf{q}; t] \tag{5}$$

The vector $^e\mathbf{q}(t)$ comprises a set of auxiliary variables of the non-linear element often required to model certain types of non-linearity as hysteresis, degradation, etc. In general, the evolution of the non-linear finite element quantities $^e\mathbf{q}(t)$ depends also on the complete state of the element, i.e.

$$^e\dot{\mathbf{q}}(t) = {}^e\mathbf{h}[{}^e\mathbf{u}, {}^e\dot{\mathbf{u}}, {}^e\mathbf{q}; t] \tag{6}$$

where $^e\mathbf{h}$ is also a non-linear vector function of the state of the element.

Applying statistical equivalent linearization approach, Equations (5) and (6) can be linearized for each non-linear finite element by the linear relation

$$\left\{ \begin{array}{c} ^e\mathbf{r}(t) \\ ^e\dot{\mathbf{q}}(t) \end{array} \right\} \doteq \begin{bmatrix} ^e\mathbf{R}_1 & ^e\mathbf{R}_2 & ^e\mathbf{R}_3 \\ ^e\mathbf{H}_1 & ^e\mathbf{H}_2 & ^e\mathbf{H}_3 \end{bmatrix} \cdot \left\{ \begin{array}{c} ^e\mathbf{u} \\ ^e\dot{\mathbf{u}} \\ ^e\mathbf{q} \end{array} \right\} \tag{7}$$

where the components $^e\mathbf{R}_1, {}^e\mathbf{R}_2, {}^e\mathbf{R}_3$ and $^e\mathbf{H}_1, {}^e\mathbf{H}_2, {}^e\mathbf{H}_3$ are sub-matrices comprising the linearization coefficients for each element. Determining the linearization coefficients for all the elements, the

above relations can be summarized in terms of global co-ordinates ($\mathbf{u}_G, \dot{\mathbf{u}}_G$ and $\mathbf{q}_G$) by the following first-order linearized system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix}\mathbf{u}_G(t)\\ \dot{\mathbf{u}}_G(t)\\ \mathbf{q}_G(t)\\ \mathbf{z}(t)\end{Bmatrix} = \mathbf{A}(t)\cdot\begin{Bmatrix}\mathbf{u}_G(t)\\ \dot{\mathbf{u}}_G(t)\\ \mathbf{q}_G(t)\\ \mathbf{z}(t)\end{Bmatrix} + \begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{w}(t)\end{Bmatrix} \tag{8}$$

The matrix $\mathbf{A}(t)$ is the so-called state-space matrix of the linearized system. The formulation of $\mathbf{A}(t)$ will be explained in detail in the next section.

3. MATHEMATICAL DERIVATION OF THE PROCEDURE

It is easily expected that the computational complexity to linearize the whole non-linear vector $\mathbf{r}(t)$ grows dramatically with increasing number of non-linear elements.

In the following, attention is directed to developing a computational procedure which allows a practical implementation of the method of equivalent linearization into FE-analysis. In the present procedure, it is suggested that the linearization of the non-linear vector $\mathbf{r}(t)$ is performed for each non-linear element (component) individually. The linearization coefficients are determined in a local co-ordinate system where number of variables (number of local co-ordinates needed for the linearization) is a minimum. Performing the linearization technique with a minimal number of co-ordinates avoids some mathematical difficulties, e.g. the problem of ill-conditioned equations which may affect the accuracy of the linearization coefficients. Consequently, the efficiency of the procedure is significantly increased.

The technique simply involves a linear co-ordinate transformation for each element from global state vector ${}^e\mathbf{y}_G$ to local state vector ${}^e\mathbf{y}_L$:

$$\begin{Bmatrix}{}^e\mathbf{u}_L\\ {}^e\dot{\mathbf{u}}_L\\ {}^e\mathbf{q}_L\end{Bmatrix} = \begin{bmatrix}\mathbf{T} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}\cdot\begin{Bmatrix}{}^e\mathbf{u}_G\\ {}^e\dot{\mathbf{u}}_G\\ {}^e\mathbf{q}_G\end{Bmatrix} \tag{9}$$

where $\mathbf{T}$ is a transformation matrix and ${}^e\mathbf{q}_L$ are the discretized auxiliary variables within each element.

The non-linear restoring forces of each finite element ${}^e\mathbf{r}_L$ can be defined in terms of the local co-ordinates as

$${}^e\mathbf{r}_L = {}^e\mathbf{g}({}^e\mathbf{y}_L) \tag{10}$$

Applying the concept of equivalent linearization, Equation (11) is replaced by the following linear relation:

$${}^e\mathbf{r}_L = {}^e\mathbf{R}_1\cdot{}^e\mathbf{u}_L + {}^e\mathbf{R}_2\cdot{}^e\dot{\mathbf{u}}_\mathbf{L} + {}^\mathbf{e}\mathbf{R}_3\cdot{}^e\mathbf{q}_L \tag{11}$$

where ${}^e\mathbf{R}_1$, ${}^e\mathbf{R}_2$ and ${}^e\mathbf{R}_3$ are sub-matrices containing the linearization coefficients for each non-linear finite element. Having calculated the linearization coefficients for each element, ${}^e\mathbf{r}_L$ is transformed back to global co-ordinates ${}^e\mathbf{r}_G$, i.e.:

$${}^e\mathbf{r}_G = \mathbf{T}^\mathrm{T}\cdot{}^e\mathbf{R}_1\cdot\mathbf{T}\cdot{}^e\mathbf{u}_G + \mathbf{T}^\mathrm{T}\cdot{}^e\mathbf{R}_2\cdot\mathbf{T}\cdot{}^e\dot{\mathbf{u}}_G + \mathbf{T}^\mathrm{T}\cdot{}^e\mathbf{R}_3\cdot{}^e\mathbf{q}_G \tag{12}$$

These global forces ${}^{e}\mathbf{r}_G$ are then added up for all the finite elements to establish the linearized vector of global forces $\mathbf{r}_G$:

$$\mathbf{r}_G = \mathbf{R}_1 \cdot \mathbf{u}_G + \mathbf{R}_2 \cdot \dot{\mathbf{u}}_G + \mathbf{R}_3 \cdot \mathbf{q}_G \tag{13}$$

It should be pointed out that, for some types of non-linearity, e.g. conservative bilinear type, the vector of auxiliary variables $\mathbf{q}$ can be omitted in describing the non-linear behaviour of the element. This means that, the only sub-matrices ${}^{e}\mathbf{R}_1$ and/or ${}^{e}\mathbf{R}_2$ are updated at each time step while ${}^{e}\mathbf{R}_3$ does not enter into the above relations.

For most non-linear types described by a vector of auxiliary variable $\mathbf{q}$, the non-linear vector of each non-linear finite element ${}^{e}\dot{\mathbf{q}}$ can be represented in local co-ordinates:

$${}^{e}\dot{\mathbf{q}}_L = {}^{e}\mathbf{h}({}^{e}\mathbf{y}_L) \tag{14}$$

where any type of non-linear restoring force model, e.g. hysteresis, can be utilized as a non-linear vector function.

Applying again the concept of equivalent linearization, Equation (14) is replaced by the linear relation

$${}^{e}\dot{\mathbf{q}}_L = {}^{e}\mathbf{H}_1 \cdot {}^{e}\mathbf{u}_L + {}^{e}\mathbf{H}_2 \cdot {}^{e}\dot{\mathbf{u}}_L + {}^{e}\mathbf{H}_3 \cdot {}^{e}\mathbf{q}_L \tag{15}$$

where ${}^{e}\mathbf{H}_1$, ${}^{e}\mathbf{H}_2$ and ${}^{e}\mathbf{H}_3$ are sub-matrices with the linearization coefficients of each non-linear finite element. After calculating the linearization coefficients for each element using its local co-ordinates, ${}^{e}\dot{\mathbf{q}}_L$ is transformed back to global co-ordinates, i.e.:

$${}^{e}\dot{\mathbf{q}}_G = {}^{e}\mathbf{H}_1 \cdot \mathbf{T} \cdot {}^{e}\mathbf{u}_G + {}^{e}\mathbf{H}_2 \cdot \mathbf{T} \cdot {}^{e}\dot{\mathbf{u}}_G + {}^{e}\mathbf{H}_3 \cdot {}^{e}\mathbf{q}_G \tag{16}$$

${}^{e}\dot{\mathbf{q}}_G$ are then added up, as done before with the local forces, to get the linearized global vector $\dot{\mathbf{q}}_G$:

$$\dot{\mathbf{q}}_G = \mathbf{H}_1 \cdot \mathbf{u}_G + \mathbf{H}_2 \cdot \dot{\mathbf{u}}_G + \mathbf{H}_3 \cdot \mathbf{q}_G \tag{17}$$

Equations (13) and (17) are used now to formulate the state-space matrix $\mathbf{A}(t)$ of the linearized system, see Equation (8):

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\hat{R}_1 & -\hat{D} - \hat{R}_2 & -\hat{R}_3 & \hat{\mathscr{E}} \\ \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathscr{F} \end{bmatrix} \tag{18}$$

where the symbol '$\hat{}$' denotes a particular matrix pre-multiplied by $\mathbf{M}^{-1}$, e.g. $\hat{\mathbf{D}} = \mathbf{M}^{-1} \cdot \mathbf{D}$, and $\mathbf{I}, \mathbf{0}$ represent the identity and zero matrices, respectively. From Equation (18), it is clear that most of sub-matrices in $\mathbf{A}(t)$ remain constant during the linearization procedure except those sub-matrices related to non-linear functions.

Finally, the evaluation of the first two moments of the linearized system can be carried out as outlined in the appendix.

4. FURTHER DEVELOPMENTS

As described in the appendix, the computation of the stochastic linearized response requires the complex left and right eigenvectors of the linearized state-space matrix $\mathbf{A}$. Since EQL is an iterative procedure, these eigenvectors need to be computed once in each iterative step. The computational

effort for this step depends heavily on the size (dimension) of the matrix $\mathbf{A}$. FE-formulations applied nowadays for structural analysis have quite often the problem of dealing with a large number of DOFs (usually more than several thousands especially for practical systems, e.g. space crafts). For a moderate dimension n, e.g. less than several hundreds, the solution of eigenvalue problems can be obtained utilizing mathematical standard packages such as LAPACK. In case of larger FE-systems, however, additional steps are needed to solve the complex eigenvalue problem in an efficient manner. Recently developed procedures, e.g. Complex Mode Synthesis [21] are capable of dealing advantageously with such larger structures. Hence, using the described methodology, there is no practical limitations with respect to the size of the FE-model.

5. SOFTWARE ASPECTS

The procedure described in the preceding sections is already embedded in an object oriented command interpreter — COSSAN [20] software environment — to be readily used for practical application.

The computational efforts can be summarized in the following programming steps:

1. Input of mass, stiffness, damping, starting mean vector and starting covariance matrix.
2. Filter matrices $\hat{\mathscr{E}}$ and $\mathscr{F}$, see Equation (18), in case of filtered white noise.
3. The local co-ordinates (displacements and velocities) for each non-linear element (device) are expressed in terms of the global system co-ordinates.
4. The restoring forces for each element are defined as a function of its local co-ordinates.
5. According to the type of non-linearity where the auxiliary variable ${}^{e}\mathbf{q}(t)$ is involved to describe the non-linear nature of the restoring forces, the non-linear function ${}^{e}\dot{\mathbf{q}}(t)$ has to be defined for each element by its local co-ordinates. This step is not essential for types of non-linearity such as conservative type, where the non-linear behaviour is described sufficiently by the restoring forces only without involving any auxiliary variable.
6. Assembling the linearized state-space matrix $\mathbf{A}$ which can be further used as an input for other commands to evaluate the stationary or non-stationary stochastic response.

6. NUMERICAL EXAMPLES

To show the applicability of the procedure presented in the preceding sections, two different MDOF-systems are analysed under bi-directional earthquake excitation. Owing to high rigidity of the floors, the dynamic behaviour of both systems may be sufficiently represented by three global (master) DOFs each floor, i.e. two translatory movements in the direction of axes 1 and 2, and a rotational movement about axis 3, respectively. For aseismic design purposes, frictional devices exhibiting one- or two-dimensional hysteresis are introduced at selected locations as shown on the plan of the two systems. The total restoring forces are considered as the sum of both linear and non-linear parts. The non-linear part is mainly provided by such frictional devices denoted by D1, D2, D3, D4 and D5 as shown in Figures 1 and 5. It is assumed that these non-linear devices are all of identical characteristics.

The earthquake-induced ground acceleration is modelled as a stationary filtered white noise. The ground acceleration components along the directions 1 and 2 are assumed to be independent and to have identical spectra which have realistic zero low-frequency asymptotes, i.e. $S_{aa}(\omega=0)=0$, and might represent stochastic earthquake excitation. Filtered white noise is obtained using the

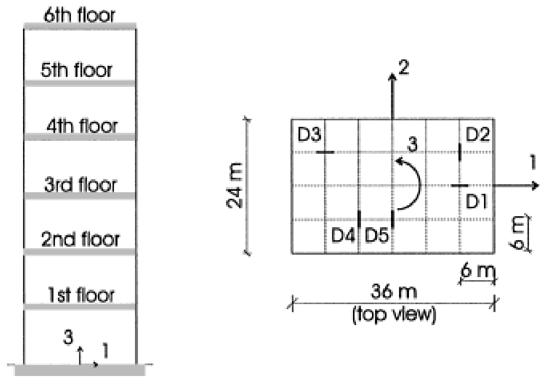

Figure 1. Non-linear six storey building.

following linear differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{Bmatrix} = \mathscr{F} \cdot \begin{Bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ w(t) \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

where the filter matrix $\mathscr{F}$ is defined as

$$\mathscr{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_{1g}^2 & -2\xi_{1g}\Omega_{1g} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ +\Omega_{1g}^2 & +2\xi_{1g}\Omega_{1g} & -\Omega_{2g}^2 & -2\xi_{2g}\Omega_{1g} \end{bmatrix} \tag{20}$$

and the ground acceleration:

$$a_g(t) = +\Omega_{1g}^2 \cdot z_1(t) + 2\xi_{1g}\Omega_{1g} \cdot z_2(t) - \Omega_{2g}^2 \cdot z_3(t) - 2\xi_{2g}\Omega_{2g} \cdot z_4(t) \tag{21}$$

where $\Omega_{1g} = 15.6$ rad/sec, $\Omega_{2g} = 1.0$ rad/sec, $\xi_{1g} = 0.6$, $\xi_{2g} = 0.995$ and $\mathbf{w}(t)$ denotes Gaussian white noise with correlation function:

$$E\{\mathbf{w}(t) \cdot \mathbf{w}^{\mathrm{T}}(t+\tau)\} = \mathbf{I_f} \cdot \delta(\tau) \tag{22}$$

$\mathbf{I_f}$ is the intensity of white noise and $\delta(\cdot)$ is Dirac delta function.

6.1. Six-storey building (one-dimensional hysteresis)

The first example considered herein (see Figure 1), consists of six rigid floors (18 global DOFs). Each component in the vector of global co-ordinates $\mathbf{u}_G$ is denoted as u_{ij} where i is the number of floor and j is the direction. The associated masses $m_1 = m_2$ and m_3 are taken as constant for all floors (1×10^6 kg and 156×10^6 kgm^2, respectively).

To investigate the effect of these devices, linear analysis is performed first for two cases, namely: without and with frictional devices, respectively. In the second case, the initial stiffness of the devices has been taken into account to evaluate the stationary response. The results of interest are

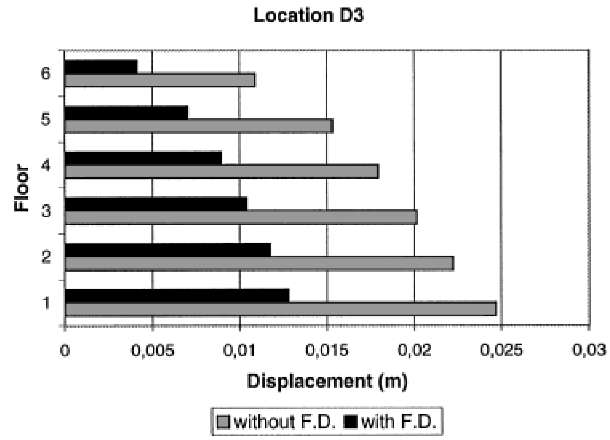

Figure 2. Standard deviation of relative displacements at D3 (Linear).

the standard deviation of the relative displacements at device D3 in both directions 1 and 2 for all floors.

For the linear analysis, the results in direction 1 at D3 are shown in Figure 2 where one can see clearly the effect of the frictional devices in decreasing the response. To start the non-linear analysis, the local co-ordinates (displacements) are formulated at the locations of the frictional devices, e.g. the transformation from global-to-local co-ordinates can be written for the first two floors as:

Floor (1):

$$\begin{aligned} u_{L_{11}} &= u_{11} \\ u_{L_{12}} &= u_{12} + 12.0 \cdot u_{13} \\ u_{L_{13}} &= u_{11} - 6.0 \cdot u_{13} \\ u_{L_{14}} &= u_{12} - 6.0 \cdot u_{13} \\ u_{L_{15}} &= u_{12} \end{aligned} \tag{23}$$

Floor (2):

$$\begin{aligned} u_{L_{21}} &= u_{21} - u_{11} \\ u_{L_{22}} &= (u_{22} - u_{12}) + 12.0 \cdot (u_{23} - u_{13}) \\ u_{L_{23}} &= (u_{21} - u_{11}) - 6.0 \cdot (u_{23} - u_{13}) \\ u_{L_{24}} &= (u_{22} - u_{12}) - 6.0 \cdot (u_{23} - u_{13}) \\ u_{L_{25}} &= (u_{22} - u_{12}) \end{aligned} \tag{24}$$

where $u_{L_{ij}}$ is the local co-ordinate at ith floor and jth device.

The local co-ordinates for displacements at all other floors can be written in the same way. The same transformation for velocities is obtained by taking the time derivatives of the displacement local co-ordinates.

The frictional devices, as shown in Figure 1 (top view), are assumed to undergo hysteresis only in one degree of freedom (one-dimensional hysteresis) for which the non-linear restoring force

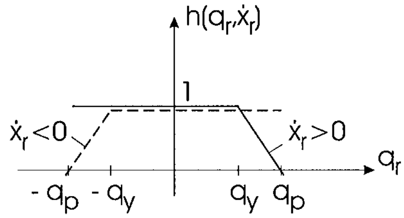

Figure 3. Hysteretic restoring force.

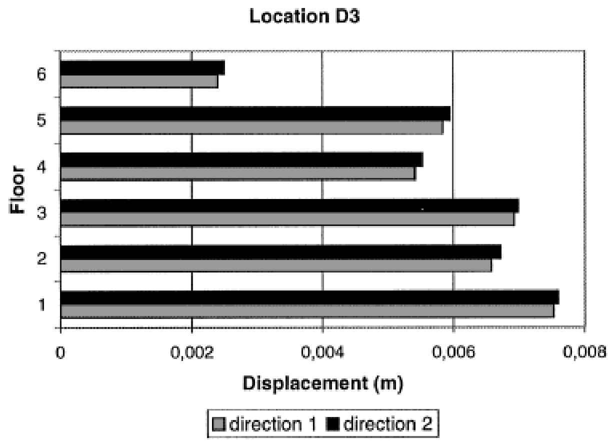

Figure 4. Standard deviation of relative displacements at D3 (proposed procedure).

law ${}^{e}r_L(t)$ can be expressed in terms of local co-ordinates as

$$ {}^{e}r_L(t) = \alpha \cdot k \cdot {}^{e}u_L + (1-\alpha) \cdot k \cdot {}^{e}q_L \tag{25} $$

where the auxiliary variables ${}^{e}q_L$ follow the differential equation:

$$ {}^{e}\dot{q}_L(t) = {}^{e}h({}^{e}q_L, {}^{e}\dot{u}_L) \cdot {}^{e}\dot{u}_L(t) \tag{26} $$

with

$$ {}^{e}h({}^{e}q_L, {}^{e}\dot{u}_L) = \begin{cases} 1 & |{}^{e}q_L| \leqslant q_y \quad \text{or} \\ & {}^{e}q_L \,.\, {}^{e}\dot{u}_L \leqslant 0 \\ (|{}^{e}q_L| - q_p)/(q_y - q_p) & |{}^{e}q_L| > q_y \quad \text{and} \\ & {}^{e}q_L \,.\, {}^{e}\dot{u}_L > 0 \end{cases} \tag{27} $$

In the above relations, ${}^{e}u_L$ represents the local displacements, k is the initial stiffness ($k = 300$ MN/m), α is a parameter (between 0 and 1) representing the remaining linear stiffness after yielding takes place. This parameter α is assumed to be 0.05. The yielding displacement q_y is taken as $q_y = 0.7 \cdot q_p$, where q_p is a plastic displacement and its value varies with respect to floors: 0.009, 0.006 and 0.003 m for floors 1 and 2, floors 3 and 4, and floors 5 and 6, respectively. The hysteretic behaviour of ${}^{e}q(t)$ is shown in Figure 3. The displacement response evaluated at D3 is shown in Figure 4.

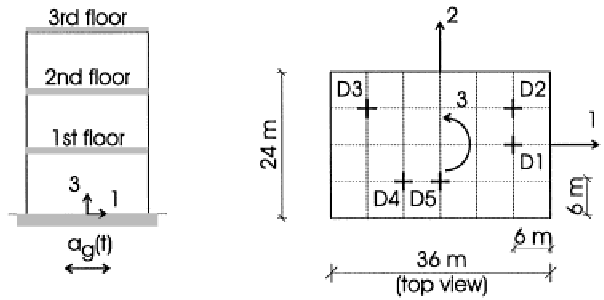

Figure 5. Non-linear three storey building.

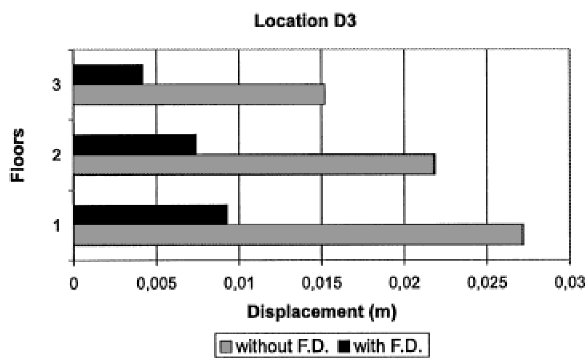

Figure 6. Standard deviation of relative displacements at D3 (Linear).

6.2. *Three-storey building (two-dimensional hysteresis)*

The second system consists of three rigid floors as shown in Figure 5. The vector of global DOFs $\mathbf{u}_G$ for the whole system is

$$\mathbf{u}_G = \{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}\}^{\mathrm{T}} \tag{28}$$

where in u_{ij}, i is the number of floor and j is the direction. The floor masses m_1, m_2, m_3 are identical to the first example. The frictional devices in this example, as shown in Figure 5, are assumed to have different characteristics. These devices exhibit a two-dimensional hysteretic type of non-linearity where the initial stiffness is constant for all devices ($k = 300$ MN/m).

Linear analysis is also performed here for two cases: without and with frictional devices, respectively. The standard deviation of the relative displacements at device D3 in direction 1 is displayed for all floors in Figure 6. The effect of the frictional devices is also clear in this example to decrease the response. To carry out the non-linear analysis, the local co-ordinates (displacements) are formulated at the locations of the frictional devices. For example, at location D1, the transformation from global-to-local co-ordinates can be written for all floors as:

Floor (1):

$$\begin{aligned} u_{L_{111}} &= u_{11} \\ u_{L_{112}} &= u_{12} + 12.0 \cdot u_{13} \end{aligned} \tag{29}$$

Floor (2):

$$\begin{aligned} u_{L_{211}} &= u_{21} - u_{11} \\ u_{L_{212}} &= (u_{22} - u_{12}) + 12.0 \cdot (u_{23} - u_{13}) \end{aligned} \tag{30}$$

Floor (3):

$$\begin{aligned} u_{L_{311}} &= u_{31} - u_{21} \\ u_{L_{312}} &= (u_{32} - u_{22}) + 12.0 \cdot (u_{33} - u_{23}) \end{aligned} \tag{31}$$

where $u_{L_{ijk}}$ is the local co-ordinate at ith floor, jth device and kth direction.

The local co-ordinates for displacements and velocities for all other devices can be written in a similar way.

Each frictional device (non-linear element) undergoes a two-dimensional hysteresis for which the non-linear restoring forces are defined in local co-ordinates as

$$\begin{Bmatrix} {}^e r_{1L} \\ {}^e r_{2L} \end{Bmatrix} = \alpha \cdot [K] \cdot \begin{Bmatrix} {}^e u_{1L} \\ {}^e u_{2L} \end{Bmatrix} + (1 - \alpha) \cdot [K] \cdot \begin{Bmatrix} {}^e q_{1L} \\ {}^e q_{2L} \end{Bmatrix} \tag{32}$$

where ${}^e u_L$ and ${}^e q_L$ are local co-ordinates, respectively, $[K]$ represents the initial stiffness and α is the ratio of the post-yielding stiffness to the initial stiffness. The auxiliary variable vector follows a differential equation defined by the state vector $({}^e q_{1L}, {}^e q_{2L}, {}^e \dot{u}_{1L}, {}^e \dot{u}_{2L})$ of each element. For any state vector $({}^e q_{1L}, {}^e q_{2L}, {}^e \dot{u}_{1L}, {}^e \dot{u}_{2L})$, one can define, on the symmetric limit (yield) surface, both radial and tangential auxiliary variables (see Figure 7):

$$\begin{aligned} {}^e q_r &= \sqrt{{}^e q_{1L}^2 + {}^e q_{2L}^2} \\ {}^e q_t &= 0.0 \end{aligned} \tag{33}$$

Also, on the same surface, the radial and tangential velocity components can be specified as

$$\begin{Bmatrix} {}^e \dot{u}_r \\ {}^e \dot{u}_t \end{Bmatrix} = \frac{1}{{}^e q_r} \begin{bmatrix} {}^e q_{1L} & {}^e q_{2L} \\ -{}^e q_{2L} & {}^e q_{1L} \end{bmatrix} \cdot \begin{Bmatrix} {}^e \dot{u}_{1L} \\ {}^e \dot{u}_{2L} \end{Bmatrix} \tag{34}$$

and associated radial and tangential derivatives are established as

$$\begin{Bmatrix} {}^e \dot{q}_r \\ {}^e \dot{q}_t \end{Bmatrix} = \begin{Bmatrix} {}^e \dot{u}_r \cdot {}^e h({}^e q_r, {}^e \dot{u}_r) \\ {}^e \dot{u}_t \end{Bmatrix} \tag{35}$$

with

$${}^e h({}^e q_r, {}^e \dot{u}_r) = \begin{cases} 1 & |{}^e q_r| \leqslant q_y \quad \text{or} \\ & {}^e q_r \cdot {}^e \dot{u}_r \leqslant 0 \\ (|{}^e q_r| - q_p)/(q_y - q_p), & |{}^e q_r| > q_y \quad \text{and} \\ & {}^e q_r \cdot {}^e \dot{u}_r > 0 \end{cases} \tag{36}$$

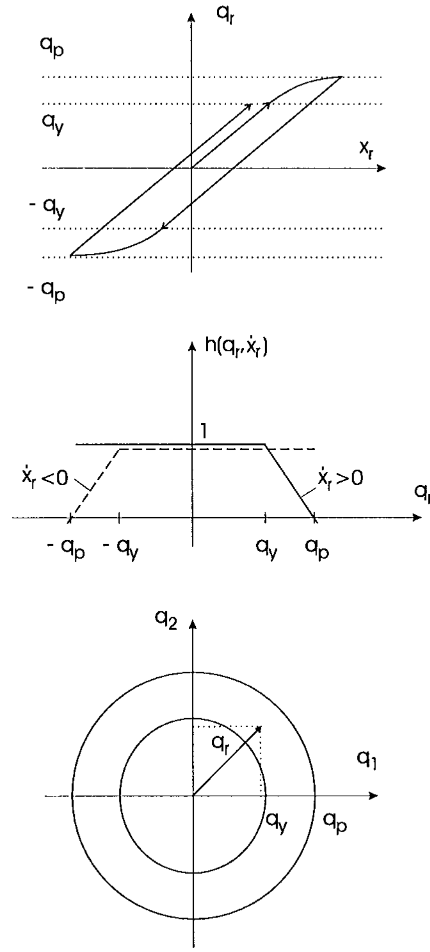

Figure 7. Two-dimensional hysteretic behavior of a single element.

Finally,

$$\left\{ \begin{array}{c} {}^e\dot{q}_{1L} \\ {}^e\dot{q}_{2L} \end{array} \right\} = \frac{1}{{}^eq_r} \begin{bmatrix} {}^eq_{1L} & -{}^eq_{2L} \\ {}^eq_{2L} & {}^eq_{1L} \end{bmatrix} \cdot \left\{ \begin{array}{c} {}^e\dot{q}_r \\ {}^e\dot{q}_t \end{array} \right\} \tag{37}$$

From equations (32)–(37), one can see that the characteristics of the hysteretic behaviour of the frictional devices are controlled by both the plastic displacement q_p and yielding displacement q_y. The variable q_p varies with respect to floors: 0.0045, 0.0030 and 0.0015 m for the first, second and third floor, respectively. The yielding displacement q_y is taken as $q_y = 0.7 \cdot q_p$. The ratio α is assumed to be 0.0.

The displacement response computed at position D3 is shown in Figure 8. In this figure, the displacement components in directions 1 and 2 are in close agreement. From Figures 4 and 8, one

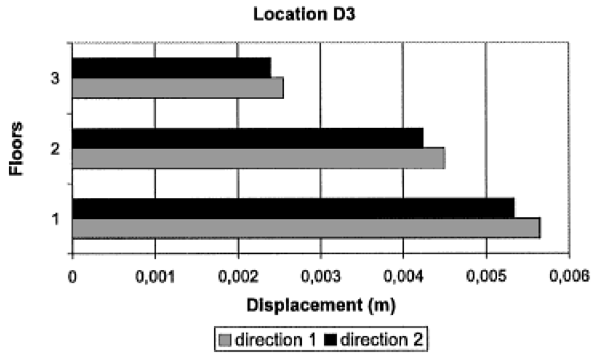

Figure 8. Standard deviation of relative displacements at D3 (proposed procedure).

can conclude that frictional devices, in the non-linear range, have the capability of suppressing the floor displacements by approximately 50 per cent of those in the linear range.

7. CONCLUSIONS

Based on the present work, the following conclusions can be drawn:

1. A computational procedure is introduced to evaluate the stochastic response of non-linear systems using the principles of statistical equivalent linearization (EQL) within FE-formulations. Each non-linear element, in the non-linear part of the system, is linearized independently applying any desired linearization criterion.
2. The linearization coefficients are obtained in a local co-ordinate system with a minimum number of variables (coordinates). This can avoid mathematical difficulties, e.g. ill-conditioned equations which may adversely affect the accuracy of the linearization coefficients. In addition, the efficiency of the procedure can be significantly enhanced.
3. The applicability and feasibility of the presented procedure have been numerically demonstrated by analysing two different MDOF-systems with frictional devices exhibiting two different types of non-linearity.
4. The proposed procedure is computationally attractive especially for arbitrary larger FE-models where available packages for non-linear finite element formulation can be used most efficiently.
5. When analysing arbitrary detailed FE-models (e.g. $>10\,000$ DOFs), methods such as Complex Mode Synthesis are suggested to couple the linearized non-linear elements with the linear structure. These methods are capable of dealing with the damping characteristics of the linearized structure.

APPENDIX A

A.1. Stochastic response evaluation

The first two moments of the response of the linearized system are evaluated here using the complex modal analysis technique [19]. The state vector $\mathbf{x}$ is transformed to the modal co-ordinates

ξ by

$$\mathbf{x} = \mathbf{\Phi} \cdot \xi \tag{A1}$$

$\mathbf{\Phi}$ is obtained by solving the eigenvalue problem

$$\mathbf{A} \cdot \mathbf{\Phi} = \mathbf{\Phi} \cdot \mathbf{\Lambda} \tag{A2}$$

and

$$\mathbf{A}^{\mathrm{T}} \cdot \mathbf{\Psi} = \mathbf{\Psi} \cdot \mathbf{\Lambda} \tag{A3}$$

with

$$\mathbf{\Lambda} = \mathbf{\Psi}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{\Phi} \tag{A4}$$

and

$$\mathbf{I} = \mathbf{\Psi}^{\mathrm{T}} \cdot \mathbf{\Phi} \tag{A5}$$

$\mathbf{\Lambda}$ is a diagonal matrix, $\mathbf{I}$ is the identity matrix, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are the normalized right and left eigenvectors. The first-order system equation in the modal co-ordinates becomes

$$\dot{\xi} = \mathbf{\Lambda} \cdot \xi + \mathbf{p}(t) \tag{A6}$$

where

$$\mathbf{p}(t) = \mathbf{\Psi}^{\mathrm{T}} \cdot [\mathbf{b}(t) + \mathbf{w}(t)] \tag{A7}$$

$\mathbf{b}(t)$ is a deterministic vector.

Having obtained the linearized state-space matrix $\mathbf{A}(t_i)$ valid within the time interval $t \in [t_i, t_{i+1}]$, the normalized right and left eigenvectors $\mathbf{\Phi}_i$ and $\mathbf{\Psi}_i$ are determined. Then, the mean vector $E\{\mathbf{x}(t_i)\}$ and the covariance matrix $\mathbf{C}_{\mathbf{xx}}(t_i)$ at time t_i are transformed into the modal co-ordinates:

$$E\{\boldsymbol{\xi}(t_i)\} = \mathbf{\Psi}_i^{\mathrm{T}} \cdot E\{\mathbf{x}(t_i)\} \tag{A8}$$

$$\mathbf{C}_{\xi\xi}(t_i) = \mathbf{\Psi}_i^{\mathrm{T}} \cdot \mathbf{C}_{\mathbf{xx}}(t_i) \cdot \mathbf{\Psi}_i^{*} \tag{A9}$$

where the symbol '$*$' denotes the conjugate complex matrix. The above quantities can be integrated to give at time $t_{i+1} = t_i + \tau_i$ for each component of the vector $E\{\xi\}$ and matrix $\mathbf{C}_{\xi\xi}$:

$$E\{\boldsymbol{\xi}_k(t_{i+1})\} = \mathrm{e}^{\lambda_k \tau_i} \cdot \left[E\{\boldsymbol{\xi}_k(t_i)\} + \int_0^{\tau_i} \mathbf{p}_k(t_i + s) \cdot \mathrm{e}^{-\lambda_k s} \, \mathrm{d}s \right] \tag{A10}$$

$$\mathbf{C}_{mn}(t_{i+1}) = \mathrm{e}^{(\lambda_m + \lambda_n^*)\tau_i} \cdot \left[\mathbf{C}_{mn}(t_i) + \int_0^{\tau_i} \mathbf{B}_{mn}(t_i + s) \cdot \mathrm{e}^{-(\lambda_m + \lambda_n^*)} \, \mathrm{d}s \right] \tag{A11}$$

with

$$\mathbf{B}_{mn}(t_i + s) = \mathbf{\Psi}_i^{\mathrm{T}} \cdot \hat{\mathbf{I}}(t_i + s) \cdot \mathbf{\Psi}_i^{*} \tag{A12}$$

where $\hat{\mathbf{I}}$ is the white noise intensity matrix. Transforming Equations (A10) and (A11) back to the original co-ordinates gives the results at time t_{i+1}:

$$E\{\mathbf{x}(t_{i+1})\} = \mathbf{\Phi}_i \cdot E\{\mathbf{x}(t_{i+1})\} \tag{A13}$$

$$\mathbf{C}_{\mathbf{xx}}(t_{i+1}) = \mathbf{\Phi}_i \cdot \mathbf{C}_{\xi\xi}(t_{i+1}) \cdot \mathbf{\Phi}_i^H \tag{A14}$$

The above relations complete the evaluation of the first two moments within a time step.

Since the covariance matrix $\mathbf{C}_{\xi\xi}$ describes the correlations of the state vector $\mathbf{x}$ in the modal coordinates, it is useful to obtain these correlations in the real space. This can be done by solving again an eigenvalue problem

$$\mathbf{C}_{\xi\xi} = \hat{\mathbf{P}} \cdot \mathbf{\Sigma} \cdot \hat{\mathbf{P}}^{\mathrm{T}} \tag{A15}$$

where $\hat{\mathbf{P}}$ and $\mathbf{\Sigma}$, are eigenvectors and eigenvalues of $\mathbf{C}_{\xi\xi}$, respectively. Since the covariance matrix $\mathbf{C}_{\xi\xi}$ is Hermitian, all the eigenvalues σ^2 in $\mathbf{\Sigma}$ are real. The covariance matrix $\mathbf{C}_{\xi\xi}$ can be expressed in terms of

$$\mathbf{C}_{\mathbf{xx}} = \mathbf{P} \cdot \mathbf{P}^{\mathrm{T}} \tag{A16}$$

with

$$\mathbf{P} = \mathbf{\Phi} \cdot \hat{\mathbf{P}} \cdot \mathrm{diag}(\sqrt{\mathbf{\Sigma}}) \tag{A17}$$

$\mathbf{P}$ is a matrix with dimensions $n \times m$, where m is the number of modes of interest. The number of non-zero columns in $\mathbf{P}$ gives the rank of the covariance matrix $\mathbf{C}_{\mathbf{xx}}$ which is less than n. This means that there is no need to include all n components of $\mathbf{x}$ during the linearization procedure for each element.

The state variables ${}^m\mathbf{x}^k$ can be generated conveniently using the relation:

$${}^m\mathbf{x}_j^k = \sum_{l=1}^{L} \mathbf{P}_{jl} \cdot {}^k r_l \tag{A18}$$

where ${}^k r_l$ are independent random numbers following a standard normal distribution N(0,1). Such sets of randomly generated independent state vectors can be used to determine the linearization coefficients by solving a least-squares problem [22].

APPENDIX B. STRUCTURAL CONTROL DEVICES

Hysteretic devices [23,24] such as friction-based devices have been used successfully to control vibrational characteristics of earthquake excitations, i.e. reduce the structural response. This is due to the fact that such systems can limit the storey shear and drift below the quasi-elastic limit.

In the design stage of a friction device, the engineer has to choose a maximum horizontal shear force envelope of the building which controls the storey drifts. The friction device can be adjusted to slide under a certain friction force. In order to obtain suitable energy dissipation at all floors, the devices have to be adjusted to slide at different force levels in each floor.

A typical formation of a friction-based device system is shown in Figure B1. The system consists of a free-standing reinforced concrete wall and the shear device connected with the top of the wall and the floor slab. A two-dimensional device is shown in Figure B2. The results of equivalent linearization and Monte Carlo Simulation show that friction devices are very effective in keeping the relative displacements and the velocities small, even for strong (stationary) earthquake excitation. Hence, the input energy is largely dissipated during only a few cycles, while the remaining structural components remain within the elastic range.

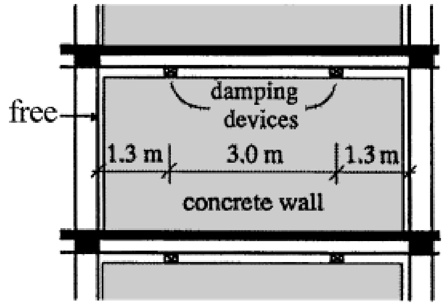

Figure B1. Friction based device system.

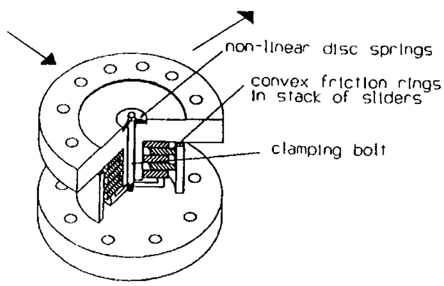

Figure B2. Two-dimensional friction device.

This fact allows a realistic modelling of the structure and consequently a reliable prediction of the response, since the modelling of reinforced components in the non-linear range (which generally causes some modelling problems) can be avoided.

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